PHYS2939 Electromagnetism
(Electrical Engineering)

Part 2:

Magnetic Fields and Materials

Maxwell's Equations and Waves.

Griffiths Chapters 5, 6, 7, sect. 9.2.

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We have now derived 4 differential equations for the electric and magnetic fields, which show interesting symmetries and asymmetries:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho \qquad \text{Gauss' law}$$

$$\nabla \times \mathbf{E} = 0 \qquad \text{(no name)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(no name)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{Ampere's law}$$

These are Maxwell's equations for electro and magnetostatics. They must be supplemented by *boundary conditions* (e.g. **E** and **B** go to zero at infinity), and the Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Electric fields are generated by charges, and field lines start on +ve charges, and end on –ve charges. However, there are no magnetic equivalents – no magnetic monopoles. How might Maxwell's equations change if there were magnetic monopoles?

Note that in current-free regions, $\nabla\times\mathbf{B} = 0$, so we may introduce a scalar potential function, which will obey Laplace's equation. This is a useful calculational tool, but it is not physical. We will now introduce the magnetic <u>vector</u> potential, which is physical.

Magnetic Vector Potential

Since the electric field is curl-less, there must exist a scalar function whose gradient is that field (this is a purely mathematical result).

By contrast, the magnetic field is divergence-less, and so there must exist a vector function, **A**, such that

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (4.9)$$

General mathematical theorem on vector fields, **F**:

The following statements are equivalent:

a) $\nabla \cdot \mathbf{F} = 0$ everywhere;

b) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of <u>surface</u> for a fixed boundary;

c) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any <u>closed</u> surface;

d) **F** is the curl of some vector, $\mathbf{F} = \nabla \times \mathbf{W}$.

(4.9) guarantees $\nabla \cdot \mathbf{B} = 0$ (because div of a curl is always zero). What does it do for Ampere's law?

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \, \mathbf{J} \qquad (4.10)$$

The electric (scalar) potential is not unique – we may add any constant (any term with a zero gradient).

This time we may add to **A** any term with zero curl (much more complex): that is to say, we may remove from **A** its divergence, and choose **A** such that

$$\nabla \cdot \mathbf{A} = 0 \qquad (4.11)$$

Now we are able to reduce (4.10) to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{4.12}$$

This is just a vector form of Poisson's equation, and may be solved (at least formally) in the same manner:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r} d\tau \tag{4.13}$$

This is still a vector, unlike the electrostatic potential, so we have gained little. However (4.13) is somewhat more straightforward than Biot-Savart.

Notes: **r** in (4.13) is the distance from a variable point (within **J**) to a fixed point where we want **A.**

In Cartesian coordinates

$$\nabla^2 \mathbf{A} = \left(\nabla^2 A_x\right)\mathbf{i} + \left(\nabla^2 A_y\right)\mathbf{j} + \left(\nabla^2 A_z\right)\mathbf{k}$$

However, in curvilinear coordinates, things can get very messy! It's best to go back to

$$\nabla^2 \mathbf{A} = \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla \times \left(\nabla \times \mathbf{A}\right)$$

Since $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$,

while $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$,

we see that **A** depends on **B** in exactly the same way that **B** depends on $\mu_0\mathbf{J}$ – that is, just like Biot-Savart. Thus we must also have

$$\mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{B} \times \hat{\mathbf{r}}}{r^2} d\tau \tag{4.14}$$

Example 1

Find the vector potential due to an infinite current:

$$(4.13) \Rightarrow \qquad \mathbf{A} = \frac{\mu_0 \mathbf{I}}{4\pi} \int \frac{dz}{R}$$

$$= \frac{\mu_0 \mathbf{I}}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{\left(z^2 + r^2\right)^{1/2}}$$

and this integral diverges at both limits. !!

There are other methods to get a sensible result.

Clearly **A** = A_z**k** – i.e. it must be parallel to **I**.

And we know $\quad \mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$

So in cylindrical coords we only require the ϕ part;

$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

The first term must be zero (above), so integrate to obtain $\quad A_z = -\frac{\mu_0}{2\pi} \ln\left(r / r_0\right)$

i.e. $\quad \mathbf{A} = -\frac{\mu_0}{2\pi} \ln\left(r / r_0\right) \mathbf{I}$

r_0 is an arbitrary constant of integration, and is the key to the solution. Note that $\mathbf{A}(r_0) = 0$.

Example 2

Vector potential due to a current sheet, $K\mathbf{i}$:

Consider a closed loop in the x-z plane:

Now $$\oint \mathbf{A}\cdot d\mathbf{l} = \int \left(\nabla\times\mathbf{A}\right)\cdot d\mathbf{a} = \int \mathbf{B}\cdot d\mathbf{a} = \Phi_B$$

(This is a general result, using Stokes' theorem.)

For the loop above, start by defining **A** = **0** at z = 0.
We found before for this current sheet that

$$\mathbf{B} = -\tfrac{1}{2}\mu_0\, K\,\mathbf{j} \qquad z > 0$$

so $$\Phi_B = \int \mathbf{B}\cdot d\mathbf{a} = -\tfrac{1}{2}\mu\ \ K\,\ell\,z$$

and $$\int \mathbf{A}\cdot d\mathbf{l} = 0 + 0 + 0 - \ell\,A(z)$$

$$\therefore\quad \mathbf{A} = \tfrac{1}{2}\mu_0\, z\,\mathbf{K}$$

Note that this loop method could have been used for the previous problem – but you can't set **A = 0** on the wire, you need an arbitrary zero point (try it!).

Similarly, you can use the differential equation method from the previous problem here – you need to use physical intuition to decide which components and derivatives are zero (important – think about it!), and then use curl in Cartesian coordinates.

Boundary Conditions

Electrostatics: ρ, **E** and V, and connecting relations.

Magnetostatics is seen now to involve **J**, **B** and **A**, and a set of relations between them (an integral expression takes us one way, and a differential equation takes us the other way – check them all.)

To complete the story we need boundary conditions: in particular, when surface currents are present.

Consider fig. 5.49 in Griffiths: a thin pill box lies across the boundary. Apply the equation

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0 \qquad \left(i.e. \quad \nabla \cdot \mathbf{B} = 0\right)$$

Hence $\qquad B_{\perp\, above} = B_{\perp\, below}$

Now consider an Amperian loop (*G*. fig. 5.50) which runs perpendicular to the surface current, **K**. The integral form of Ampere's law now gives

$$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B_{\parallel\, above} - B_{\parallel\, below}\right)\ell = \mu_0\, I_{enc} = \mu_0\, K\, \ell$$

Clearly the component of **B** parallel to **K** doesn't change. All these results can be combined into one:

$$\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0 \left(\mathbf{K} \times \hat{\mathbf{n}}\right) \qquad (4.15)$$

where $\hat{\mathbf{n}}$ is a unit vector pointing "up" (i.e. from 'below' to 'above').

As with the electrostatic potential, *V*, the magnetic vector potential, **A**, is continuous across a boundary, although its derivative is not (from **B** = ∇ × **A**).

Example 3

Find the vector potential for an infinite solenoid with n turns per unit length, radius R, and current I:

We know the magnetic field for a solenoid:

Inside $\mathbf{B} = \mu_0 \, n I \, \hat{\mathbf{z}}$

Outside $\mathbf{B} = 0$.

Let us use the loop method from Example 2

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi_B$$

Take the loop to be of radius r from the solenoid axis:

Inside we have

$$\oint \mathbf{A} \cdot d\mathbf{l} = 2\pi r \, A = \int \mathbf{B} \cdot d\mathbf{a} = (\mu_0 n \, I)(\pi \, r^2)$$

$$\mathbf{A} = \tfrac{1}{2} \mu_0 n I \, r \, \hat{\phi} \quad \textit{for } r < R$$

Now outside; note that we only have **B** out to R, so

$$\oint \mathbf{A} \cdot d\mathbf{l} = 2\pi r \, A = \int \mathbf{B} \cdot d\mathbf{a} = (\mu_0 n \, I)(\pi \, R^2)$$

$$\mathbf{A} = \tfrac{1}{2} \mu_0 n I \, \frac{R^2}{r} \hat{\phi} \quad \textit{for } r > R$$

Note that outside, **A** is non-zero, even though **B** = 0!

Could we do this problem via the DE method?

Inside we certainly can, provide we realise that **A** will only have a ϕ component (and **B** only a z component). Outside, we would need to use boundary conditions: **A** is continuous at $r = R$ (check this).

The Magnetic Dipole

A magnetic dipole is a small current loop with (in principle) arbitrary shape. Just as with the electric dipole, the magnetic dipole has useful applications, so we wish to compute both the vector potential, and the magnetic field, of such a current loop.

This task is distinctly more difficult than for the electric case, so we won't do the job in full, but only sketch the method and give the results.

The vector potential at an arbitrary point is given by

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l}}{r'}$$

The geometry is given in fig. 5.51 from Griffiths for an arbitrary shaped loop.

After various expansions and approximations (as for the electric dipole, but more complex), for the case of a <u>circular loop</u> of radius *a*, we obtain

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{I \pi a^2}{r^2} \sin\theta \, \hat{\phi} \qquad (4.17)$$

This result can be generalized in several ways.

We start by defining the (magnetic) dipole moment as a product of the current times the loop area, and give it an 'upwards' (vector) direction:

$$\mathbf{m} = I \pi a^2 \, \hat{\mathbf{k}} \qquad (4.18)$$

and then further generalize this to non-circular loops with the definition:

$$\mathbf{m} = \tfrac{1}{2} I \oint \mathbf{r} \times d\mathbf{l} \tag{4.19}$$

This allows us to re-write eq.(4.17) as

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \tag{4.17'}$$

We now find **B** from $\nabla \times$ **A** in spherical coords:

$$B_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\mu_0}{4\pi} \frac{m}{r^2} \sin\theta \right)$$

$$= \frac{\mu_0}{4\pi} \frac{2m}{r^3} \cos\theta$$

$$B_\theta = -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\mu_0}{4\pi} \frac{m}{r^2} \sin\theta \right)$$

$$= \frac{\mu_0}{4\pi} \frac{m}{r^3} \sin\theta$$

$$B_\phi = 0 \, .$$

i.e. $$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{m}{r^3} \left(2\cos\theta \, \hat{\mathbf{r}} + \sin\theta \, \hat{\phi} \right) \tag{4.20}$$

This result is valid for distances large compared with the charactistic size of the current loop.

Magnetic dipoles experience both torques and forces when placed in magnetic fields. In a uniform field, the net force on a dipole is zero. However, in a non-uniform field, **B**, the net force can be shown to be

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) = \mathbf{m} \times (\nabla \times \mathbf{B}) + (\mathbf{m} \cdot \nabla)\mathbf{B}$$

$$= (\mathbf{m} \cdot \nabla)\mathbf{B} \tag{4.22}$$

since $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = 0$, since **B** is an external field.

Torques on Magnetic Dipoles

Consider a magnetic dipole in the form of a square loop of side *s*, and carrying a current *I*. Let **B** be in the *z* direction, and **m** be at an angle θ to **B**:

B will exert forces on all 4 sides. The forces on the two sides in the *y*-*z* plane will cancel (they will tend to stretch, or compress, the loop – assume it's rigid). The forces on the two sides in the *x* direction will be equal and opposite (net force is zero), but they are not co-linear. Thus they exert a torque on the loop:

$$\mathbf{N} = s\,F\sin\theta\,\hat{\mathbf{i}}$$

and the force (magnitude) on each side is

$$F = I\,s\,B$$

$$\therefore \quad \mathbf{N} = I\,s^2\,B\sin\theta\,\hat{\mathbf{i}} = m\,B\sin\theta\,\hat{\mathbf{i}}$$

i.e. $$\mathbf{N} = \mathbf{m} \times \mathbf{B} \tag{4.21}$$

While derived for a square loop, this result applies to any shape loop – i.e. any magnetic dipole.

Note that this torque will tend to align **m** parallel to the field **B**, and hence add its field to **B**. This torque accounts for paramagnetism (next lecture).