

PHYS2939 Electromagnetism

(Electrical Engineering)

Part 2:

Magnetic Fields and Materials

Maxwell's Equations and Waves.

Griffiths Chapters 5, 6, 7, sect. 9.1, 2.

A/Prof. Michael Box (room 102B, OMB)

Lecture 11

Electromagnetic Waves

Plane Waves

Lecture 12

Energy in Electromagnetic Waves

Electromagnetic Waves in Dielectrics

“Revision”: read sections 9.1.1 and 9.1.2

Electromagnetic Waves

Consider Maxwell's equations in vacuum, where there are no free charges and no free currents. In this case, the equations reduce to

$$\begin{aligned} i) \quad \nabla \cdot \mathbf{E} &= 0 & (iii) \quad \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ ii) \quad \nabla \cdot \mathbf{B} &= 0 & (iv) \quad \nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

This is a set of coupled, first order, partial differential equations. Take the curl of (iii), and use (iv):

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \\ &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \end{aligned}$$

which, using (i), reduces to

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (6.32 \text{ a})$$

Taking the curl of (iv), and using (iii) and (ii) gives

$$\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \quad (6.32 \text{ b})$$

Thus, each Cartesian component satisfies the equation

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \quad (6.33)$$

This is the (three dimensional) wave equation.

In one dimension (for ease of study), this becomes

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

which has a general solution of the form

$$f(x, t) = g(x - vt)$$

where g is any function of the single variable, $x-vt$. This represents a wave, travelling to the right, with a speed v (Griffiths has the proof, but it's trivial).

From this we may conclude that equations (6.32) represent the propagation of electromagnetic waves in free space (vacuum) with the speed

$$\begin{aligned} v &= (\epsilon_0 \mu_0)^{-1/2} = (8.854 \times 10^{-12} \times 4\pi \times 10^{-7})^{-1/2} \\ &= (1.112 \times 10^{-17})^{-1/2} = 3 \times 10^8 \text{ m/s} \end{aligned} \quad (6.34)$$

But this is just the speed of light, c !! Thus light is an electromagnetic wave – an alternating electric and magnetic field. Maxwell's amazing (and controversial) discovery was later "confirmed" by Hertz. Notes:

1. ϵ_0 and μ_0 are determined by electrostatic and magnetostatic measurements, yet their values determine the speed of light (in vacuum).
2. The role of Maxwell's displacement current is crucial: without it we wouldn't have the wave equation.
3. A changing electric field is creating a changing magnetic field, which is creating a changing electric field, which is...

Plane Waves

Before we go any further, we need a reminder of the best way to write/manipulate a plane wave, namely *complex notation*: Consider the function

$$\tilde{f}(x,t) = \tilde{A} \exp[i(kx - \omega t)]$$

where the \sim indicates a complex function/number. From Euler's formula,

$$e^{i\theta} = \cos \theta + i \sin \theta$$

we see that the real part of our complex function is

$$f(x,t) = \text{Re}[\tilde{A} e^{i(kx - \omega t)}] = A \sin(kx - \omega t + \delta)$$

where the (irrelevant) phase constant δ , takes care of the 'rest' of A . Thus, we can write a plane wave either in sine form, or complex exponential form (and then the real part is taken as understood).

Finally remember that

$$k = 2\pi / \lambda \quad \text{and} \quad \omega = 2\pi \nu$$

so that $v = \lambda \nu$ equates to $v = \omega / k$

The simplest solution to the wave equation is a plane wave of unique wavelength and frequency: that implies "colour", so such waves are called *monochromatic*. Consider one such plane wave, which we write most conveniently as:

$$\mathbf{E} = \mathbf{E}_0 e^{i(kx - \omega t)} \quad (6.35)$$

This wave is propagating in the x -direction. If we now assume that \mathbf{E} is constant over planes in y and z (a ‘plane wave’), that is \mathbf{E} depends only on x , then

$$\mathbf{E}_0 = \mathbf{E}_0(x)$$

This electric field must still obey all of Maxwell’s equations, and hence it must be divergenceless:

$$\nabla \cdot \mathbf{E}_0 \equiv \frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} + \frac{\partial E_{0z}}{\partial z} = \frac{\partial E_{0x}}{\partial x} = 0$$

(The other two derivatives are zero because \mathbf{E}_0 only depends on x , by our assumption of a plane wave.)

That is, \mathbf{E}_0 may have a y and/or z component, but not an x component. So electromagnetic waves must be *transverse*, and may be *polarized* in either of the two directions perpendicular to the propagation direction:

$$\begin{aligned} \mathbf{E}_0(x) &= E_1(x)\mathbf{j} + E_2(x)\mathbf{k} \\ &= E_0(x)\cos\theta\mathbf{j} + E_0(x)\sin\theta\mathbf{k} \end{aligned} \quad (6.36)$$

What about the magnetic field? Faraday’s law gives its relation to the electric field. Let’s assume

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(kx-\omega t)} \\ \mathbf{B} &= \mathbf{B}_0 e^{i(kx-\omega t)} \end{aligned} \quad (6.35')$$

Then, making use of (6.36)

$$\nabla \times \mathbf{E} = \nabla \times E_0(\cos\theta\mathbf{j} + \sin\theta\mathbf{k})e^{i(kx-\omega t)}$$

(since the only thing to differentiate is the exponential)

$$= i k E_0 (-\sin \theta \mathbf{j} + \cos \theta \mathbf{k}) e^{i(kx - \omega t)}$$

(check this to make sure you can do it!)

while
$$-\frac{\partial \mathbf{B}}{\partial t} = i \omega \mathbf{B}_0 e^{i(kx - \omega t)}$$

Equating these (as per Faraday's law)

$$i \omega \mathbf{B}_0 = i k E_0 (-\sin \theta \mathbf{j} + \cos \theta \mathbf{k})$$

i.e.
$$\mathbf{B}_0 = \frac{k}{\omega} E_0 (-\sin \theta \mathbf{j} + \cos \theta \mathbf{k})$$

$$= \frac{k}{\omega} (\mathbf{i} \times \mathbf{E}_0) \tag{6.37}$$

Thus \mathbf{E} and \mathbf{B} are in phase, and mutually perpendicular (and both are transverse to the propagation direction).

Finally, their amplitudes are related via

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0 \tag{6.38}$$

using
$$v \equiv c = \omega / k$$

Note that the 'fact' of B being so much smaller than E is primarily a consequence of the SI units we are using. In Gaussian units, they would actually be equal!

Real EM waves are a mixture of wavelengths/frequencies, directions, polarizations and phases.

Energy in Electromagnetic Waves

Poynting proved a very important, general theorem for energy flows and densities in electromagnetic fields:

$$\frac{dW}{dt} = \int_v (\mathbf{E} \cdot \mathbf{J}) d\tau = -\frac{d}{dt} \int_v (w_e + w_m) d\tau - \oint_S \mathbf{S} \cdot d\mathbf{a} \quad (6.39)$$

The first term – $\mathbf{E} \cdot \mathbf{J}$ – represents the *loss* of electromagnetic energy (within a given volume) due to ohmic heating (we have used Ohm's law).

The second term, where

$$w_e = \frac{1}{2} \epsilon_0 E^2 \quad \text{and} \quad w_m = \frac{1}{2} \mu_0^{-1} B^2 \quad (6.40)$$

represents the electric and magnetic field energies.

Finally the third term: we define the *Poynting* vector

$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \quad (6.41)$$

\mathbf{S} represents the flow of energy, per unit time, per unit area (the intensity), across the surface of our volume.

The Poynting theorem is the basic statement of energy conservation in electrodynamics. Basically, the energy stored in electromagnetic fields can flow out of (or into) a given region, or lost in the form of heat.

From (6.38) we see that

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \varepsilon_0 E^2$$

so the electric and magnetic energies are equal, and

$$w = w_e + w_m = \varepsilon_0 E^2 = \varepsilon_0 E_0 \cos^2(kx - \omega t)$$

Finally, averaging over one cycle, we obtain

$$\bar{w} = \frac{1}{2} \varepsilon_0 E_0^2 \quad (6.42)$$

Now apply (6.41) to (6.37):

$$\mathbf{S} = \frac{1}{\mu_0} \frac{1}{c} E^2 \mathbf{i} = c \varepsilon_0 E_0^2 \cos^2(kx - \omega t) \mathbf{i}$$

Once again we need to time-average to obtain the beam intensity:

$$\mathbf{I} = \bar{\mathbf{S}} = \frac{1}{2} c \varepsilon_0 E_0^2 \mathbf{i} = c \bar{w} \mathbf{i} \quad (6.43)$$

It is also possible to show that electromagnetic waves carry momentum (and even angular momentum).

The momentum density is $\bar{\mathbf{p}} = \bar{\mathbf{S}} / c^2$.

Exercise: At the Earth's surface, sunlight has a typical intensity of 1000 Wm^{-2} . Find the amplitude, E_0 .

Answer: Approximately 160 N/C.

Electromagnetic Waves in Dielectrics

This time consider electromagnetic fields inside matter, but again with no free charges or currents. In such a case, Maxwell's equations reduce to

$$i) \quad \nabla \cdot \mathbf{D} = 0 \quad (iii) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$ii) \quad \nabla \cdot \mathbf{B} = 0 \quad (iv) \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

If the medium is *linear*, then

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mathbf{H} = \mu \mathbf{B}$$

and *homogeneous*, these equations reduce to

$$i) \quad \nabla \cdot \mathbf{E} = 0 \quad (iii) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$ii) \quad \nabla \cdot \mathbf{B} = 0 \quad (iv) \quad \nabla \times \mathbf{B} = \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

This leads to an equation identical to (6.32), but with $\mu_0 \varepsilon_0$ replaced with $\mu \varepsilon$. Thus the speed of light here is

$$v = 1/\sqrt{\varepsilon \mu}$$

But in optics, we define the *refractive index*, n , by

$$v = c/n$$

$$\therefore n = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} \quad (6.44)$$

so that refractive index is an electromagnetic property!

(Note that since $\mu \approx \mu_0$ for paramagnets, refractive index is mainly related to the dielectric constant.)

We may now use Maxwell's equations, plus the all-important boundary conditions, to derive all the well-known *optics* results relating to reflection and refraction at an interface, plus a good deal more:

1. Angle of incidence equals angle of reflection;
2. Snell's law of refraction;
3. Any phase changes on reflection;
4. The fraction of energy reflected versus the fraction refracted, for each polarization state, as a function of the angle of incidence (Fresnel equns.);
5. And hence, the Brewster angle.

(Read it all in Griffiths.)

Just one quick point: the dielectric "constant" is actually not a constant, but varies with frequency. Its value at 'optical' frequencies may be very different from that measured in electrostatics.

Maxwell's equations also tell us that

6. An accelerating electric charge radiates energy in the form of electromagnetic waves;
7. So atoms will disintegrate in 10^{-8} seconds!
8. Woops – good bye.