

# **PHYS2939 Electromagnetism**

## **(Electrical Engineering)**

Part 2:

Magnetic Fields and Materials

Maxwell's Equations and Waves.

Griffiths Chapters 5, 6, 7, sect. 9.2.

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Lecture 1.

Magnetic Forces

Biot-Savart Law

Properties of the Magnetic Field

Lecture 2.

Currents and continuity

Divergence and curl of B

Applications

# Magnetic Forces

A charged particle with charge  $q$ , moving with a velocity  $\mathbf{v}$ , in a magnetic field  $\mathbf{B}$ , experiences a force

$$\mathbf{F}_m = q \mathbf{v} \times \mathbf{B} \quad (4.1)$$

If there is also an electric field present, the total, or Lorentz force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (4.1')$$

Because of the nature of the force law, magnetic forces do no work:

$$dW_m = \mathbf{F}_m \cdot d\mathbf{l} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt = 0$$

For this reason, magnetic fields do not admit a potential energy function.

The units of  $\mathbf{B}$  are teslas (T):

$$1 \text{ tesla} = 1 \text{ N} / (\text{Am})$$

Read Examples 1 and 2 on pages 205-207.

## Biot-Savart Law

Magnetostatics => fields caused by ‘steady’ currents. However, it turns out that 50 Hz is still ‘steady’.

Note that such currents cannot just start and end anywhere, but must be in closed loops (which may be closed at infinity).

For such a current, the field it produces is given by the Biot-Savart law:

$$\mathbf{B}(P) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\mathbf{r}}}{r^2} d\ell = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \quad (4.2)$$

where  $P$  is the point where we wish to know  $\mathbf{B}$ , and  $\mathbf{r}$  is a vector from a point on (within) the current to the point  $P$ .  $\mu_0$  is a constant – the permeability of free space – and its value is

$$\mu_0 = 4\pi \times 10^{-7} \text{ N A}^{-2}$$

Read *and understand* Examples 5 and 6. Some of these results are important, so we list them below:

## Examples:

1. The magnetic field due to an infinite straight line current:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi} \quad (4.3)$$

where  $r$  is the distance from the current, and  $\hat{\phi}$  is the unit vector (right hand rule). From this result we may obtain the force between two parallel currents:

$$\frac{F}{\ell} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

2. The magnetic field a distance  $z$  above the centre of a loop radius  $R$ :

$$\mathbf{B} = \frac{1}{2} \mu_0 I \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

3. The magnetic field on the axis of a disc of radius  $R$ , with a surface charge density  $\sigma$ , rotating with angular velocity  $\omega$ . Treat it as a set of rings of radius  $dr$ , each with a current  $dI$ , where

$$dI = \sigma \omega r dr$$

$$\therefore B = \frac{\mu_0 \sigma \omega}{2} \left\{ \frac{R^2 + 2z^2}{(R^2 + z^2)^{1/2}} - 2z \right\}$$

(Can you figure out how this behaves as  $z \rightarrow \infty$  ?)

## Currents – Equation of Continuity

Current density,  $\mathbf{J}$ , may be regarded as the flow of charge density,  $\rho$ , with a velocity,  $\mathbf{v}$ :

$$\mathbf{J} = \rho \mathbf{v}$$

The total current crossing (out of) any closed surface,  $S$ , is given by

$$I = \oint_S \mathbf{J} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{J}) d\tau$$

using the divergence theorem.

Since electric charge is a universally conserved quantity, this charge leaving the volume must be accounted for by a decrease in total charge, which we may represent as an integral over the charge density:

$$\int (\nabla \cdot \mathbf{J}) d\tau = -\frac{d}{dt} \int \rho d\tau = -\int \frac{\partial \rho}{\partial t} d\tau$$

Hence 
$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Note that in the case of steady currents, this implies

$$\nabla \cdot \mathbf{J} = 0$$

Finally we will need to use surface currents:

$$\mathbf{K} = \sigma \mathbf{v} = d\mathbf{I} / d\ell_{\perp}$$

i.e. the current per unit length perpendicular to flow.

## Properties of the Magnetic Field

Magnetic fields (and hence field lines) are very different from electric fields (and field lines):

The electric field due to a single, point charge can be represented by radial lines (outward or inward).

*The magnetic field due to an infinite, straight current can be represented by a set of concentric circles.*

This difference leads to radical differences in the divergence and curl of the two fields.

Start by calculating the line integral of  $\mathbf{B}$  around one of the closed field lines:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi r} d\ell = \frac{\mu_0 I}{2\pi r} \oint d\ell = \mu_0 I$$

This is independent of the distance,  $r$ .

In fact, the fact that we used a circular path is also irrelevant. Consider a more general path, with

$$\begin{aligned} d\mathbf{l} &= dr \hat{\mathbf{r}} + r d\phi \hat{\phi} + dz \hat{\mathbf{z}} \\ \oint \mathbf{B} \cdot d\mathbf{l} &= \frac{\mu_0 I}{2\pi} \oint \left( \frac{1}{r} \hat{\phi} \right) \cdot (\hat{\mathbf{r}} dr + r \hat{\phi} d\phi + \hat{\mathbf{z}} dz) \\ &= \frac{\mu_0 I}{2\pi} \oint \frac{1}{r} r d\phi = \mu_0 I \end{aligned}$$

Thus the only thing important about the loop is that it enclosed a current  $I$ . If there are several currents, then their fields just add (superposition), and so must their respective line integrals. Thus we conclude

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc} \quad (4.4)$$

This is Ampere's law in integral form.

We can express the enclosed current in terms of a surface integral over a current density:

$$I_{enc} = \int \mathbf{J} \cdot d\mathbf{a}$$

Line integrals and surface integrals may be related via Stokes theorem: apply this to the left side of (4.4)

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\therefore \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (4.5)$$

This is Ampere's law in differential (equation) form, and relates changes in  $\mathbf{B}$  to the current density at a point in space – at least for infinite, straight currents.

## Divergence and Curl of $\mathbf{B}$

Generalize Biot-Savart law to current density:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^2} d\tau \quad (4.6)$$

with  $\mathbf{B} = \mathbf{B}(x, y, z)$

$$\mathbf{J} = \mathbf{J}(x', y', z')$$

$$\mathbf{r} = (x - x')\mathbf{i} + (y - y')\mathbf{j} + (z - z')\mathbf{k}$$

and  $d\tau = dx' dy' dz'$

We will be taking the divergence and curl with respect to the unprimed coordinates (of course).

Firstly the divergence:

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

But 
$$\nabla \cdot \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) = \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left( \nabla \times \frac{\hat{\mathbf{r}}}{r^2} \right)$$

The first term is zero because  $\mathbf{J}$  depends only on the primed coordinates: the second term is identically zero because it involves ‘cross derivatives’ of the radius vector. So we conclude that

$$\nabla \cdot \mathbf{B} = 0. \quad (4.7)$$

Now for the curl:

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau \quad (4.8)$$

Now

$$\begin{aligned} \nabla \times \left( \mathbf{J} \times \frac{\hat{\mathbf{r}}}{r^2} \right) &= \left( \frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla \right) \mathbf{J} - (\mathbf{J} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2} \\ &\quad + \mathbf{J} \left( \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) - \frac{\hat{\mathbf{r}}}{r^2} (\nabla \cdot \mathbf{J}) \\ &= \mathbf{J} \left( \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2} \end{aligned}$$

(The first and fourth terms were zero since  $\mathbf{J}$  is not a function of the unprimed coordinates.)

Look at the remaining terms:

Now

$$-(\mathbf{J} \cdot \nabla) \frac{\hat{\mathbf{r}}}{r^2} = (\mathbf{J} \cdot \nabla') \frac{\hat{\mathbf{r}}}{r^2}$$

(i.e. swapping the differentiation over to the primed coordinates, which causes a sign change.)

Take this one component at a time:

$$(\mathbf{J} \cdot \nabla') \frac{x - x'}{r^3} = \nabla' \cdot \left( \frac{x - x'}{r^3} \mathbf{J} \right) - \left( \frac{x - x'}{r^3} \right) (\nabla' \cdot \mathbf{J})$$

But for steady currents (magnetostatics)

$$\nabla' \cdot \mathbf{J} = 0$$

Thus, for each component, we have

$$-\left(\mathbf{J} \cdot \nabla\right) \frac{\hat{\mathbf{r}}}{r^2} \Big|_x = \nabla' \cdot \left( \frac{x-x'}{r^3} \mathbf{J} \right)$$

Now substitute this back into the integral of (4.8), and use the divergence theorem:

$$\int_{vol} \nabla' \cdot \left( \frac{x-x'}{r^3} \mathbf{J} \right) d\tau = \oint \frac{x-x'}{r^3} \mathbf{J} \cdot d\mathbf{a}$$

Now if we make this volume big enough, we will eventually reach a point where  $\mathbf{J} = \mathbf{0}$ . (In the case of the infinite straight wire, the denominator will make the integral go to zero.) Thus we conclude that this term is zero – for all 3 components.

Only one term remains (see sect. 1.5.3; p50):

Since 
$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi \delta(\mathbf{r})$$

this term becomes

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta(\mathbf{r} - \mathbf{r}') d\tau' = \mu_0 \mathbf{J} \quad (4.5)$$

in agreement with the result from the infinite straight wire.

## Applications

Now read *and understand* Examples 5.7, 8, 9 and 10.

7. The standard result for a straight line current.

8. An infinite uniform surface current,  $\mathbf{K} = K \mathbf{i}$ , in the  $x$ - $y$  plane: find  $\mathbf{B}$ . Use a rectangular Amperian loop in the  $y$ - $z$  plane, crossing the surface: result

$$\mathbf{B} = \begin{cases} \frac{1}{2} \mu_0 K \mathbf{j} & z < 0 \\ -\frac{1}{2} \mu_0 K \mathbf{j} & z > 0 \end{cases}$$

9. An infinitely long solenoid; radius  $R$ ,  $n$  current loops per unit length, carrying a current  $I$ : find  $\mathbf{B}$ . Use a rectangular Amperian loop, inside/outside the solenoid: result

Inside  $\mathbf{B} = \mu_0 n I \hat{\mathbf{z}}$

Outside  $\mathbf{B} = 0$ .

### Example 5.10

A toroidal solenoid, with  $N$  turns (total): find  $\mathbf{B}$ .

It is ‘intuitively obvious’ (but Griffiths goes to great pains to actually prove it), that the magnetic field,  $\mathbf{B}$ , will be circumferential. It is then very easy to apply Ampere’s law, using a circumferential loop of radius  $s$ , somewhere inside the toroid: then we see that

$$2\pi s B = \mu_0 I_{enc} = \mu_0 N I$$

Hence, since  $\mathbf{B}$  has only a  $\phi$  component,

$$\mathbf{B}(s) = \frac{\mu_0 N I}{2\pi s} \hat{\phi}$$

Outside the toroid, the field is, of course, zero.