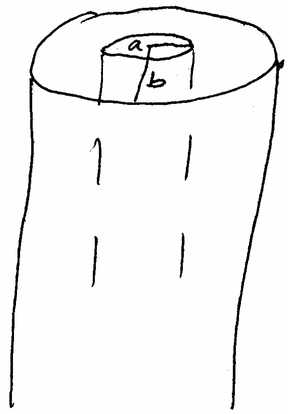
Sheet 4

1

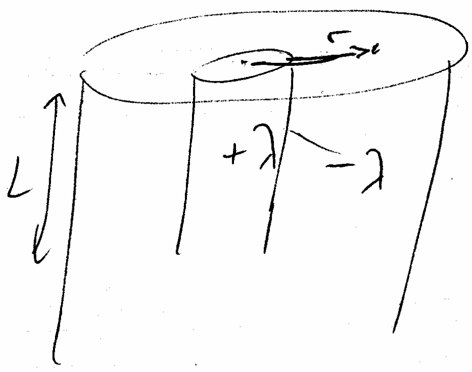


2 coaxial metal cylinders

Capacitance per unit length = ?

$$Q \rightarrow E \rightarrow V \rightarrow C$$

So, add some charge



$$E(r) = ?$$

$$\oint \underline{E} \cdot d\underline{a} = \frac{Q_{encl}}{\varepsilon_0}$$

$$E \cdot 2\pi r L = \frac{\lambda L}{\varepsilon_0}$$

$$E(r) = \frac{\lambda}{2\pi \varepsilon_0 r}$$

$$|\Delta V| = \left| \int_a^b \underline{E} \cdot d\underline{r} \right|$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \left| \int_a^b \frac{dr}{r} \right|$$

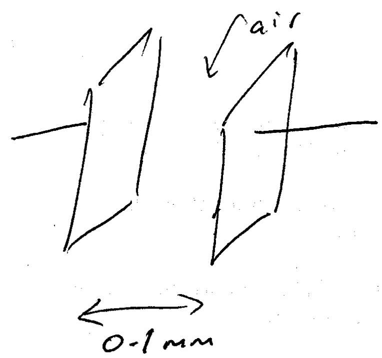
$\cancel{t}$ $$|\Delta V| = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

$$Q = CV$$

$$\therefore C = \frac{Q}{V} = \frac{\lambda L}{V}$$

$$\therefore \frac{C}{L} = \frac{\lambda}{V} = 2\pi\varepsilon_0 / \ln(b/a)$$

2/



200 V

$A_{plate} = 5 \text{ cm}^2$

a) charge density on plates

$$C = \varepsilon_0 \frac{A}{d}$$

$$q = CV = \frac{\varepsilon_0 A V}{d}$$

$$\sigma = (\rho) = \frac{q}{A} = \frac{\varepsilon_0 V}{d} = 1.77 \times 10^{-5} \text{ Cm}^{-2}$$

b) force pulling the plates together

$$U = \frac{1}{2} C V^2$$

$$= \frac{1}{2} \frac{\varepsilon_0 A}{x} V^2$$

$$F = -\frac{\partial U}{\partial x} = \frac{1}{2} \frac{\varepsilon_0 A V^2}{x^2} = \frac{1}{2} \varepsilon_0 A E^2$$

$$E = \frac{V}{x}$$

2 b

$$\therefore \quad F = \frac{1}{2}\varepsilon_0 A E^2$$

$$= \frac{1}{2} \times 8.85\times10^{-12} \times 5\times10^{-4} \times \left(\frac{200}{0.1\times10^{-3}}\right)^2$$

$$= 8.85\times10^{-3}\ N$$

c) total energy stored in the field

$$U = \frac{1}{2} C V^2$$

$$= \frac{1}{2}\frac{\varepsilon_0 A}{x} V^2$$

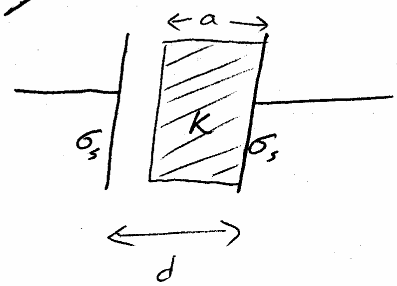
$$= \frac{1}{2} \frac{\times 8.85\times10^{-12} \times 5\times10^{-4} \times 200^2}{0.1\times10^{-3}}$$

$$= 8.85\times10^{-7}\ J$$

or $U = Fx = 8.85\times10^{-3} \times 0.1\times10^{-3}$

$$= 8.85\times10^{-7}\ J$$

3/



$\Delta V = ?$

without dielectric $E = \frac{\sigma_s}{\varepsilon_0}$

with dielectric $D = \sigma_s$

and $E = \frac{D}{k\varepsilon_0} = \frac{\sigma_s}{k\varepsilon_0}$

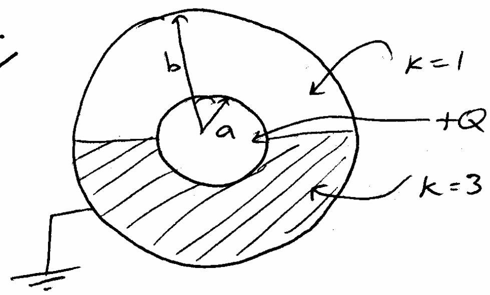
$$\therefore |\Delta V| = \left| -\int_a^b \underline{E} \cdot d\underline{l} \right|$$

$$= (d-a) \times E_{outside} + (a \times E_{inside})$$

$$= \frac{\sigma_s (d-a)}{\varepsilon_0} + \frac{a\sigma_s}{k\varepsilon_0}$$

$$= \frac{\sigma_s}{\varepsilon_0}\left[d - a + \frac{a}{k}\right]$$

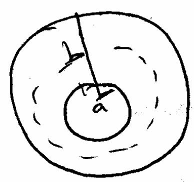
4/



the inner and outer conducting spheres form equipotential surfaces!

$\therefore E(r)$ is independent of θ, ϕ

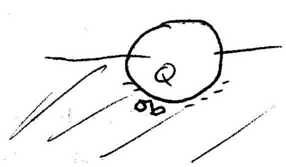
Gauss' law



$$4\pi r^2 E = \frac{Q_{encl}}{\varepsilon_0}$$

$Q_{encl} = Q$ on inner sphere

$- q$ induced on dielectric



$$= Q_{free} - \sigma_b \cdot \frac{1}{2} \cdot 4\pi a^2$$

↑ lower $\frac{1}{2}$ where the dielectric is

-4/

$$\therefore E(r) = \frac{Q - 2\pi a^2 \sigma_b}{4\pi\varepsilon_0 r^2}$$

So, now we need σ_b!

$$\sigma_b = \underset{\sim}{P} \cdot \underset{\sim}{\hat{n}} = P$$

(sphere $\underline{\hat{n}} = \underline{\hat{r}}$ and $\underline{P} = P\,\underline{\hat{r}}$)

$$\underset{\sim}{P} = \varepsilon_0 \chi_e \underset{\sim}{E}$$

$k=3 \quad \therefore \quad \chi_e = k - 1 = 2$

$$\therefore \sigma_b = 2\varepsilon_0 E(a)$$

$$= 2\varepsilon_0 \left[\frac{Q - 2\pi a^2 \sigma_b}{4\pi\varepsilon_0 a^2} \right]$$

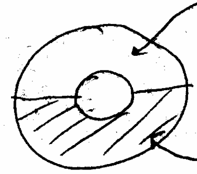
$$= \frac{Q}{2\pi a^2} - \sigma_b$$

$$\therefore \sigma_b = \frac{Q}{4\pi a^2}$$

$$\therefore \underline{E}(r) = \frac{Q - \frac{Q}{2}}{4\pi\varepsilon_0 r^2} = \frac{Q}{8\pi\varepsilon_0 r^2}\,\hat{\underline{r}}$$

#4 $\underset{\sim}{D} = ?$

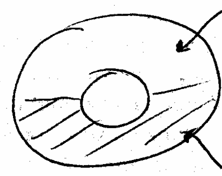
$$\underset{\sim}{D} = \kappa \varepsilon_0 \underset{\sim}{E}$$



$\underset{\sim}{D} = \varepsilon_0 \underset{\sim}{E} = Q/8\pi r^2 \; \hat{r}$

$\underset{\sim}{D} = 3\varepsilon_0 \underset{\sim}{E} = \dfrac{3Q}{8\pi r^2} \; \hat{r}$

$\underset{\sim}{P} = ?$ $\qquad \underset{\sim}{P} = \varepsilon_0 \chi_e \underset{\sim}{E}$



$\underset{\sim}{P} = 0 \quad (\kappa = 1 \therefore \chi = \kappa - 1 = 0)$

$\underset{\sim}{P} = 2\varepsilon_0 \underset{\sim}{E} \qquad (\kappa = 3 \therefore \chi_e = 2)$

b/ Free charge densities on the <u>inner</u> conductor

equipotential $\therefore \sigma_{total}$ is uniform

4

$$\sigma_b \text{ in dielectric} = -\frac{Q}{4\pi a^2}$$

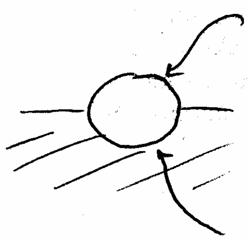
$$Q_{total} = Q + 2\pi a^2 \sigma_b$$

$$= Q - \frac{Q}{2} = \frac{Q}{2}$$

$$\therefore \sigma_{total} = \frac{Q_{total}}{4\pi a^2}$$

$$= \frac{Q}{8\pi a^2}$$

$$\sigma_s = \sigma_{free} = Q/8\pi a^2$$



$$\sigma_s = \sigma_{free} - \sigma_{bound}$$

$$= \frac{Q}{8\pi a^2} + \frac{Q}{4\pi a^2}$$

$$= 3Q/8\pi a^2$$

c/ ΔV between spheres

$$E = \frac{Q}{8\pi\varepsilon_0 r^2}$$

$$\therefore |\Delta V| = \left|\int_a^b \underline{E}\cdot d\underline{r}\right| = \frac{Q}{8\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{8\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

#5 $\dashv\vdash^{C_0}$ is placed in a tank. Then

(i) a battery (V) is connected to $\dashv\vdash$

(ii) tank is filled with oil (K)

(iii) $\dashv\vdash$ disconnected and oil is removed

Energy stored in $\dashv\vdash$ at each stage $= ?$

(i) $U = \frac{1}{2} C_0 V^2$

(ii) $U \rightarrow \frac{1}{2} K C_0 V^2$

(iii) Q must remain constant, V changes

$$Q = K C_0 V = C_0 V_{final}$$

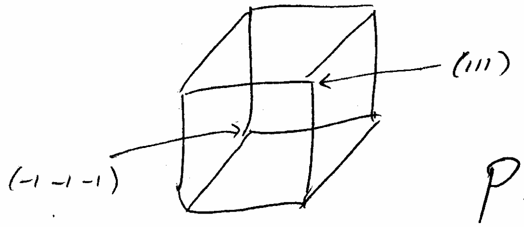
$$\therefore V_{final} = K V_{initial}$$

$$\therefore U_{final} = \frac{1}{2} C_0 V_{final}^2$$

$$= \frac{1}{2} K^2 C_0 V_{initial}^2$$

$$= \frac{1}{2} K^2 C_0 V^2$$

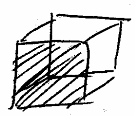
6/ cube of side 2m (6)



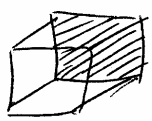
$$\underset{\sim}{P} = x\,\underline{\hat{i}} + y\,\underline{\hat{j}} \quad Cm^{-2}$$

a) σ_b on each face = ?

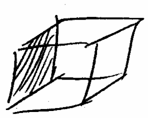
$$\sigma_b = \underline{P} \cdot \underline{\hat{n}}$$

front face  $x = 1$ $\quad \underline{\hat{n}} = \underline{\hat{i}}$

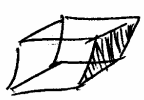
$$\therefore \sigma_b = 1\,\underline{\hat{i}} \cdot \underline{\hat{i}} = 1$$

back  $x = -1$ $\quad \underline{\hat{n}} = -\underline{\hat{i}}$

$$\sigma_b = -1\,\underline{\hat{i}} \cdot -\underline{\hat{i}} = 1$$

LHS  $y = -1$ $\quad \underline{\hat{n}} = -\underline{\hat{j}}$

$$\sigma_b = 1$$

RHS  $y = 1$ $\quad \underline{\hat{n}} = +\underline{\hat{j}}$

$$\therefore \sigma_b = 1$$

6 top and bottom faces $\sigma_b = 0$

S_1 $\because$ $\hat{\underline{n}} = \pm \hat{\underline{k}}$ and

$\underline{P}$ has no $\hat{\underline{k}}$ component

b) volume bound charge inside cube

$$\rho_b = -\underline{\nabla} \cdot \underline{P}$$

$$= -\left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0)\right]$$

$$= -2 \ C m^{-3}$$

Check neutrality condition

$$V_{cube} = 2^3 = 8 \ m^3$$

c) and d)

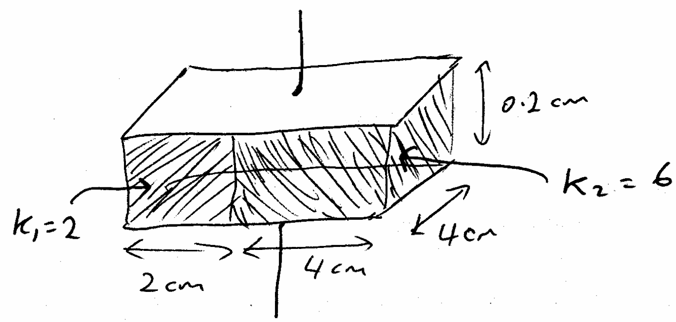
$$Q_{inside} = -2 \times 8 = -16 \ C$$

$$A_{surfaces} = 4 \text{ faces} \times 2^2 = 16 \ m^2$$

(top 2 faces have $\sigma_b = 0$)

$$\therefore Q_{surfaces} = 1 \times 16 = +16 \ C$$

#7



100 V applied.

The plates are equipotentials

$$\Delta V = -E\int dl = E l$$

so $$E = \frac{\Delta V}{l} = \frac{100}{0.2\times10^{-2}} = 5\times10^{4}\ \mathrm{V m^{-1}}$$

in both regions

c). $$D_1 = \varepsilon_1 E_1 = k_1 \varepsilon_0 E_1$$

$$= k_1 \varepsilon_0 E$$

$$= 2\times\varepsilon_0\times5\times10^{4} = 10^{5}\,\varepsilon_0$$

$$D_2 = \varepsilon_2 E_2 = k_2 \varepsilon_0 E$$

$$= 6\times\varepsilon_0\times5\times10^{4} = 3\times10^{5}\,\varepsilon_0$$

#7

e) $P_1 = \varepsilon_0 \chi_1 E$

$\chi_1 = k_1 - 1 = 1$

$\therefore P_1 = 5\times10^4\ \varepsilon_0$

f) $P_2 = \varepsilon_0 \chi_2 E \qquad \chi_2 = k_2 - 1 = 5$

$= 2.5\times10^5\ \varepsilon_0$

g) $C_{total} = ?$

2 capacitors in parallel

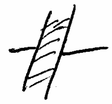
$\therefore C = C_1 + C_2$

$$C_1 = \frac{k_1 \varepsilon_0 A_1}{d_1} = \frac{2\varepsilon_0 \times 8\times10^{-4}}{0.2\times10^{-2}} = 0.8\ \varepsilon_0\ F$$

$$C_2 = \frac{k_2 \varepsilon_0 A_2}{d_2} = \frac{6\varepsilon_0 \times 16\times10^{-4}}{0.2\times10^{-2}} = 4.8\ \varepsilon_0\ F$$

$\therefore C = 5.6\ \varepsilon_0\ F = 5\times10^{-11}\ F = 50\ pF$

8/



$C = 1\mu F$ charged to 3000V then disconnected.

$k = 3$.

Remove the dielectric.

Δ stored energy = ?

$$C = kC_0 = 3C_0 = 1\mu F$$

$$\therefore C_0 = \frac{1}{3}\mu F$$

$$U_{initial} = \frac{1}{2}CV^2 = \frac{1}{2} \times 1 \times 10^{-6} \times 3000^2$$

$$= 4.5\,J$$

Remove dielectric

$C \to C_0$ $\qquad Q$ is constant

$$Q = CV_{init} = C_0 V_{final} = k.C_0 V_{init}$$

$$\therefore V_{final} = kV_{init} = 9000\,V$$

$$U = \frac{1}{2} C_0 V_f^2 = 13.5\,J$$

So $\Delta U = 9\,J$ $\uparrow$

Q9/ coaxial cable $r_a = 0.8$ cm

$r_b = 3$ cm

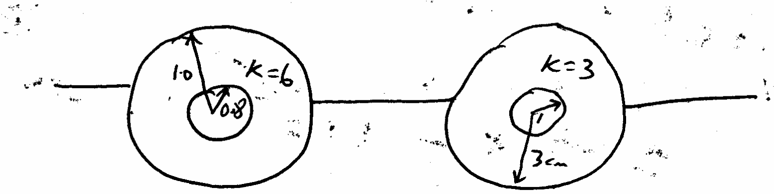
2 concentric layers of dielectrics

& $K_{inner} = 6.0$ for $0.8 - 1.0$ cm

$K_{outer} = 3.0$ for $1.0 - 3.0$ cm

PD across conductors = 12.5 kV

Hint treat as 2 cylindrical $\dashv\vdash$ in series



$Q \to E \to V \to C$

$$E = \frac{\lambda}{2\pi\varepsilon_0 k r}$$

↖ !

See Q5 in last section

So $\Delta V = -\int \underset{\sim}{E} \cdot d\underset{\sim}{r}$

#89

$$\Delta V_I = \frac{\lambda}{2\pi\varepsilon_0 \cdot 6} \int_{0.8}^{1.0\,cm} \frac{dr}{r}$$

$$= \frac{\lambda}{12\pi\varepsilon_0} \ln\left(\frac{1}{0.8}\right)$$

$$Q = CV = \lambda L$$

$$\therefore \frac{C_I}{L} = \frac{\lambda}{V} = 12\pi\varepsilon_0 / \ln\left(\frac{1}{0.8}\right)$$

Similarly $\quad \dfrac{C_{II}}{L} = \dfrac{6\pi\varepsilon_0}{\ln\left(\frac{3.0}{1.0}\right)} \qquad \kappa = 3$

$$\frac{C_I}{L} = 1.495\,nF\,m^{-1}$$

$$\frac{C_{II}}{L} = 0.152\,nF\,m^{-1}$$

series $\therefore \quad \dfrac{1}{C_{eff}} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$

$$\rightarrow C_{eff}/L = 138\,pF/m$$

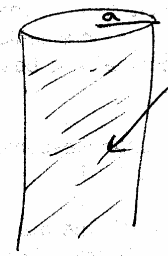
~~8~~ 9 b) Surface charge density on the inner conductor

We can first calculate λ (linear)

$$Q = CV \quad \text{so} \quad \frac{Q}{L} = \lambda = \frac{C}{L} V$$

$$\frac{C}{L} = 138\,pF \quad \text{and} \quad V = 12.5\,kV$$

$$\therefore \lambda = 1.725 \times 10^{-6}\ C m^{-1}$$



$$\sigma\ C m^{-2} = \frac{\lambda}{2\pi a}$$

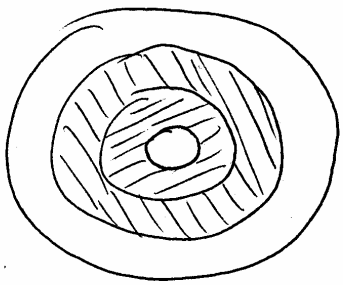
$$= \frac{1.725 \times 10^{-6}}{2\pi \times 0.8 \times 10^{-2}} = 3.43 \times 10^{-5}\ C m^{-2}$$

c) $$E_{max} = \frac{\lambda}{2\pi \varepsilon_0 k\, r_{min}}$$

$$E_{max,1} = \frac{1.725 \times 10^{-6}}{12\pi \varepsilon_0 \times 0.8 \times 10^{-2}} = 6.46 \times 10^{5}\ V m^{-1}$$

$$E_{max,2} = \frac{1.725 \times 10^{-6}}{6\pi \varepsilon_0 \times 1 \times 10^{-2}} = 1.03 \times 10^{6}\ V m^{-1}$$

~~8~~9 Could also use $\underset{\sim}{D}$



$$\oint \underset{\sim}{D} \cdot d\underset{\sim}{a} = Q_{free,encl.}$$

$\therefore$ Cylindrical G surface

$$D \cdot 2\pi r L = Q_{free,encl} = \lambda L$$

$$\therefore D = \frac{\lambda}{2\pi r}$$

$$\underset{\sim}{D} = k\varepsilon_0 \underset{\sim}{E}$$

$$\therefore E_1 = \frac{D_1}{k_1 \varepsilon_0} = \frac{\lambda}{12\pi\varepsilon_0 r} \qquad k_1 = 6$$

$$E_2 = \frac{D_2}{k_2\varepsilon_0} = \frac{\lambda}{6\pi\varepsilon_0 r} \qquad k_2 = 3$$

then ΔV as before