1. $\underline{a} = 2\hat{\underline{i}} + 4\hat{\underline{j}} + 5\hat{\underline{k}}$

$\underline{b} = 2\hat{\underline{i}} + 4\hat{\underline{j}} - 4\hat{\underline{k}}$

$$\underline{a} \cdot \underline{b} = (2 \times 2) + (4 \times 4) + (5 \times -4) = 0$$

$$\therefore \quad \underline{a} \perp \underline{b}$$

---

2. $\underline{a} = \hat{\underline{i}} + 2\hat{\underline{j}} + 3\hat{\underline{k}}$

$\underline{b} = 3\hat{\underline{i}} + 2\hat{\underline{j}} + \hat{\underline{k}}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = -4\hat{\underline{i}} + 8\hat{\underline{j}} - 4\hat{\underline{k}}$$

$$|\underline{a} \times \underline{b}| = \sqrt{[(-4)^2 + (8)^2 + (-4)^2]} = \sqrt{96}$$

$\therefore$ unit vector in dir^ $\underline{a} \times \underline{b}$ is

$$\frac{-4\hat{\underline{i}} + 8\hat{\underline{j}} - 4\hat{\underline{k}}}{\sqrt{96}} = \frac{-\hat{\underline{i}} + 2\hat{\underline{j}} - \hat{\underline{k}}}{\sqrt{6}}$$

3

$$\underline{a} = \hat{\underline{i}} + \hat{\underline{j}} + \hat{\underline{k}}$$

$$\underline{b} = \hat{\underline{i}} + 2\hat{\underline{j}} + 3\hat{\underline{k}}$$

$$\underline{c} = 3\hat{\underline{i}} + 2\hat{\underline{j}} + \hat{\underline{k}}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}$$

$$(\underline{a}\times\underline{b})\cdot\underline{c} = (\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}})\cdot(3\hat{\underline{i}} + 2\hat{\underline{j}} + \hat{\underline{k}})$$

$$= 0$$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 0$$

4. $\underline{\nabla} \cdot (f \underline{a})$

$$= \left(\hat{\underline{i}} \frac{\partial}{\partial x} + \hat{\underline{j}} \frac{\partial}{\partial y} + \hat{\underline{k}} \frac{\partial}{\partial z} \right) \cdot \left(f a_x \hat{\underline{i}} + f a_y \hat{\underline{j}} + f a_z \hat{\underline{k}} \right)$$

$$= \frac{\partial (f a_x)}{\partial x} + \frac{\partial (f a_y)}{\partial y} + \frac{\partial (f a_z)}{\partial z}$$

$$= f \frac{\partial a_x}{\partial x} + a_x \frac{\partial f}{\partial x} + f \frac{\partial a_y}{\partial y} + a_y \frac{\partial f}{\partial y} + f \frac{\partial a_z}{\partial z} + a_z \frac{\partial f}{\partial z}$$

$$= f \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) + a_x \frac{\partial f}{\partial x} + a_y \frac{\partial f}{\partial y} + a_z \frac{\partial f}{\partial z}$$

$$= f (\underline{\nabla} \cdot \underline{a}) + \underline{a} \cdot (\underline{\nabla} f)$$

5

$$\underline{\nabla} \times (\underline{\nabla} f)$$

$$= \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{\underline{i}} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right)$$

$$- \hat{\underline{j}} \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right)$$

$$+ \hat{\underline{k}} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$= 0$ since $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ etc

6. Gradient Theorem

$$T = x^2 + 4xy + 2yz^3$$

$$\int_a^b \underline{\nabla} T \cdot d\underline{l} = T(b) - T(a)$$

Initial point (a) = (0 0 0)
Final " (b) = (1 1 1)

$$\therefore \int_a^b \underline{\nabla} T \cdot d\underline{l} = T(111) - T(000) = 7$$

Path ① (000) → (100) → (110) → (111)

$$\underline{\nabla} T = \underline{\hat{i}} \frac{\partial T}{\partial x} + \underline{\hat{j}} \frac{\partial T}{\partial y} + \underline{\hat{k}} \frac{\partial T}{\partial z}$$

$$= \underline{\hat{i}}(2x + 4y) + \underline{\hat{j}}(4x + 2z^3) + \underline{\hat{k}}(6yz^2)$$

$$d\underline{l} = \underline{\hat{i}}\, dx + \underline{\hat{j}}\, dy + \underline{\hat{k}}\, dz$$

6 1st bit, dy and $dz = 0$

$$\underline{\nabla} T \cdot d\underline{l} = (2x + 4y)\,dx$$

$$= 2x\,dx \qquad \because y = 0$$

$$\int_{1\text{st bit}} \underline{\nabla} T \cdot d\underline{l} = \int_0^1 2x\,dx = 1$$

2nd bit $(1\,0\,0) \rightarrow (1\,1\,0)$

$x = 1 \qquad dx = 0$

$z = 0 \qquad dz = 0$

$y : \quad 0 \rightarrow 1$

$$\underline{\nabla} T \cdot d\underline{l} = (4x + 2z^3)\,dy$$

$$= 4x\,dy \qquad \because z = 0$$

$$= 4\,dy \qquad \because x = 1$$

$$\int_{\substack{2\text{nd} \\ \text{bit}}} \underline{\nabla} T \cdot d\underline{l} = \int_0^1 4\,dy = 4$$

6/ 3rd bit $(110) \rightarrow (111)$

$$x = 1 \qquad dx = 0$$
$$y = 1 \qquad dy = 0$$
$$z : 0 \rightarrow 1$$

$$\underline{\nabla} T \cdot d\underline{l} = 6yz^2 dz = 6z^2 dz$$

$$\int_{\substack{3rd\\bit}} \underline{\nabla} T \cdot d\underline{l} = \int_0^1 6z^2 dz = 2$$

$\therefore$ Total Path

$$\int \underline{\nabla} T \cdot d\underline{l} = 1 + 4 + 2 = 7$$

Same procedure for path 2

$$\underbrace{(000) \rightarrow (001)}_{=0} \underbrace{\rightarrow (011)}_{=2} \underbrace{\rightarrow (111)}_{=5}$$

$$\text{Total} = 0 + 2 + 5 = 7$$

7. Divergence Theorem

$$\int_V (\underline{\nabla} \cdot \underline{A})\, d\tau = \oint_S \underline{A} \cdot d\underline{a}$$

$$\underline{A} = xy\,\hat{\underline{i}} + 2yz\,\hat{\underline{j}} + 3zx\,\hat{\underline{k}}$$

Cube, side length = 2

$$\underline{\nabla} \cdot \underline{A} = y + 2z + 3x$$

$$\int_V (\underline{\nabla} \cdot \underline{A})\, d\tau = \int_0^2 \int_0^2 \int_0^2 (y + 2z + 3x)\, dx\, dy\, dz$$

$\int dz$ part : $\rightarrow$ $$\int_0^2 \int_0^2 \left[yz + z^2 + 3xz \right]_0^2 dx\, dy$$

$$= \int_0^2 \int_0^2 (2y + 4 + 6x)\, dx\, dy$$

~~#~~ 7/

$\int dy$ part:

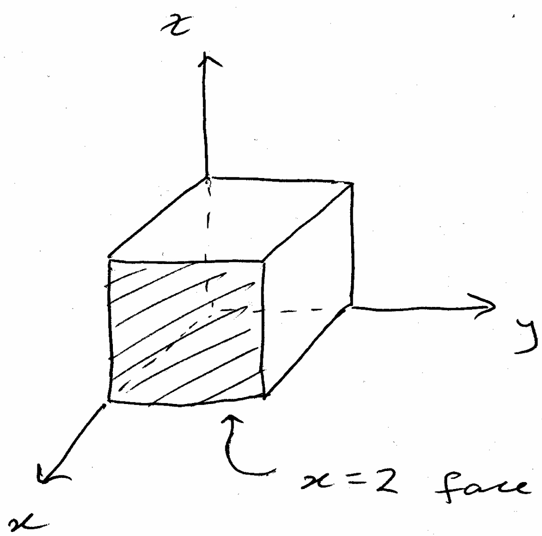
$$\int_{x=0}^{2} \left[y^2 + 4y + 6xy \right]_0^2 \, dx$$

$$= \int_{x=0}^{2} (4 + 8 + 12x) \, dx$$

$$= \int_{x=0}^{2} (12 + 12x) \, dx$$

$$= \left[12x + 6x^2 \right]_0^2 = 48$$

Now do the surface integral



$x = 2$ face

$d\underline{a} = \cancel{dxdy}\ \underline{\hat{i}}$

$dy\,dz$

7/

$$\underline{A}\cdot d\underline{a} = xy\,dy\,dz$$
$$= 2y\,dy\,dz \qquad (x=2)$$

$\therefore \int \underline{A}\cdot d\underline{a}$ for this face is

$$\int_{y=0}^{2}\int_{z=0}^{2} 2y\,dy\,dz = 8$$

other faces:

$x=0$	$\int\cdots$	$=0$
$y=2$		16
$y=0$		0
$z=2$		24
$z=0$		0

$$\therefore \oint_S \underline{A}\cdot d\underline{a} = 8+16+24 = 48$$

$$= \int_V (\underline{\nabla}\cdot\underline{A})\,d\tau$$

8 Stokes' Theorem
(Curl Theorem)

$$\int_S (\underline{\nabla} \times \underline{A}) \cdot d\underline{a} = \oint_L \underline{A} \cdot d\underline{l}$$

$$\underline{A} = (2xz + 3y^2)\,\hat{\underline{j}} + 4yz^2\,\hat{\underline{k}}$$

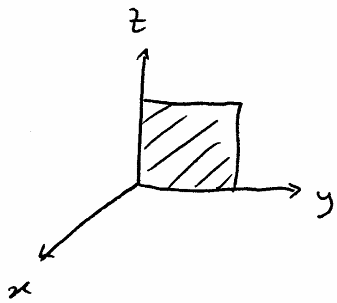
+ square, side = 1, corners (000) ~~too~~ (011)

$$\underline{\nabla} \times \underline{A} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (2xz+3y^2) & (4yz^2) \end{vmatrix}$$

$$= \hat{\underline{i}}\,(4z^2 - 2x) + \hat{\underline{k}}\,(2z)$$

$$= \hat{\underline{i}}\,(4z^2) + \hat{\underline{k}}\,(2z) \qquad \because x = 0$$

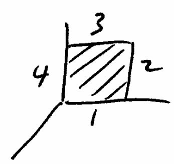
8



$$d\underline{a} = dy\,dz\,\hat{\underline{i}}$$

$$\int_S (\underline{\nabla} \times \underline{A}) \cdot d\underline{a} = \int_0^1 \int_0^1 (4z^2 \hat{\underline{i}} + 2z\hat{\underline{k}}) \cdot (dy\,dz\,\hat{\underline{i}})$$

$$= \int_0^1 \int_0^1 4z^2\,dy\,dz = 4/3$$



edges 1–4

1. $(000) \rightarrow (010)$

$$\int \underline{A} \cdot d\underline{l} = \int (2xz + 3y^2)\hat{\underline{j}} \cdot dy\,\hat{\underline{j}}$$

$$= \int_0^1 3y^2\,dy = 1$$

2. $(010) \rightarrow (011)$ $\quad d\underline{l} = dz\,\hat{\underline{k}}$

$$\int_0^1 4yz^2\,dz = \int_0^1 4z^2\,dz = 4/3$$

8/

3. $(0 1 1) \rightarrow (0 0 1)$ $\quad d\underline{l} = dy\,\underline{\hat{j}}$

$$\int_1^0 3y^2\,dy = -1$$

4. $(0 0 1) \rightarrow (0 0 0)$

$$= 0 \quad \because y = 0$$

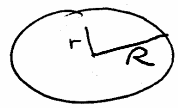
$$\therefore \oint_L \underline{A} \cdot d\underline{l} = +1 + \frac{4}{3} - 1 + 0$$

$$= \frac{4}{3}$$

$$= \int_S (\underline{\nabla} \times \underline{A}) \cdot d\underline{a}$$

9 Surface and Volume Integrals

sphere



$$\rho = 4Ar \quad Cm^{-3} \quad (A = \text{constant})$$

$Q_{total} = ?$

$$Q = \int dq = \int \rho \, d\tau$$

$$= 4A \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} r \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 8\pi A \int_{\theta} \int_{r} r^3 \sin\theta \, dr \, d\theta$$

$$= 8\pi A \int_{r=0}^{R} r^3 \Big[-\cos\theta \Big]_0^{\pi} \, dr$$

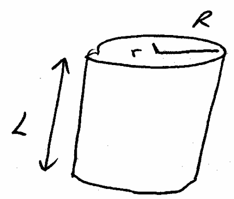
$$= 16\pi A \int r^3 dr$$

$$= 16\pi A \cdot \frac{1}{4} \left[r^4 \right]_0^R$$

$$= 4\pi A R^4 \quad \text{Coulomb}$$

Units of A $=$ Cm^{-4}

10/ Surf. & Vol. $\int$



$\rho = \frac{A}{r}$ Cm^{-3} in central bit

and $\rho = Br^2$ Cm^{-2} on each flat end

$Q_{total} = Q_{central} + 2\,Q_{end}$

Cylindrical coords. $d\tau = r\,dr\,d\phi\,dz$

$$Q_{end} = \int_{\phi=0}^{2\pi}\int_{r=0}^{R} Br^2 \cdot \underbrace{r\,dr\,d\phi}_{\text{d. area } \therefore \text{ no } z}$$

$$= 2\pi B \left[\frac{1}{4}r^4\right]_0^R$$

$$= \frac{\pi B R^4}{2}$$

$$Q_{central} = \int_{z=-L/2}^{+L/2}\int_{\phi=0}^{2\pi}\int_{r=0}^{R} \frac{A}{r} \cdot r\,dr\,d\phi\,dz$$

$$= A \cdot R \cdot 2\pi \cdot L$$

$\therefore Q_{total} = 2\pi ARL + \pi BR^4$ Cool.

$[A] = Cm^{-2}$ $[B] = Cm^{-4}$