A/ Prof. C. Ulrich

Electromagnetism PHYS2050

Example: Cycloid

Calculate the length of the path of a single point on a wheel, when the wheel is rolling one full round:

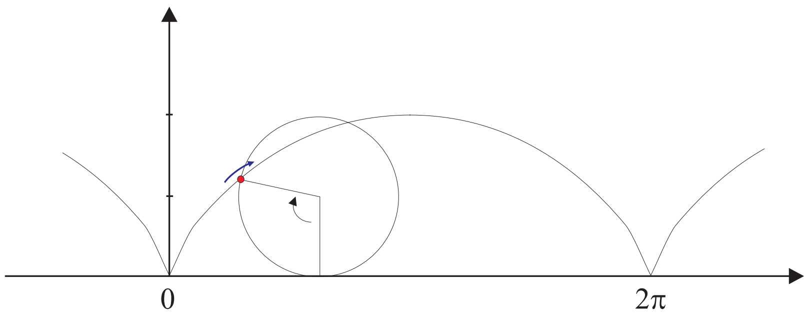

Figure 1: cycloid: path of a single point on a rolling wheel.

$$\vec{x}(t) = \underbrace{r\begin{pmatrix} t \\ 1 \end{pmatrix}}_{\substack{horizontal \\ movement}} + \underbrace{r\begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix}}_{\substack{circular \\ movement}}$$

$$\vec{x}(t) = \begin{pmatrix} r(t-\sin(t)) \\ r(1-\cos(t)) \end{pmatrix} \qquad \dot{\vec{x}}(t) = \begin{pmatrix} r(1-\cos(t)) \\ r\ \sin(t) \end{pmatrix}$$

$$\begin{aligned} ds &= \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\ dt \\ &= r\sqrt{(1-\cos(t))^2 + \sin^2(t)}\ dt \\ &= r\sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}\ dt \\ &= r\sqrt{2}\ \sqrt{1-\cos(t)}\ dt \\ &= r\sqrt{2}\ \sqrt{2\sin^2(t/2)}\ dt \\ &= 2r\ \sin(t/2)\ dt \end{aligned}$$

Theferore, the length of the path of a point on a wheel , i.e. the path of a cycloid is:

$$\begin{aligned} Z &= \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\ dt = 2r \int_0^{2\pi} \sin(t/2)\ dt \\ &= 2r\ \left[-2\cos(t/2)\right]\Big|_0^{2\pi} = 8\ r \end{aligned}$$

This corresponds to 4 times the diameter of the wheel. Note that π is canceled out.