PART V

Path integration method.
Propagator.

Path integration method

Eigenfunction of position operator

$$\psi_{x_a} \equiv |x_a\rangle = \delta(x - x_a)$$

$$\hat{x}\,|x_a\rangle = x_a\,|x_a\rangle$$

Normalization

$$\langle \psi_{x_a} | \psi_{x_b} \rangle = \int \delta(x-x_a)\,\delta(x-x_b)\,dx = \delta(x_a - x_b)$$

Note that this normalization is unusual: $\langle \psi_{x_a} | \psi_{x_a} \rangle = \infty$.

Nevertheless this is a convinient normalization since if one has a usual smooth wave function $\varphi(x)$ and takes projection on the state $|x_a\rangle$

$$\langle x_a | \varphi \rangle = \int \delta(x - x_a)\,\varphi(x)\,dx = \varphi(x_a)$$

then the procedure gives a usual probability amplitude.

Closer relation (completeness relation)

$$\sum_{x_a} |x_a\rangle\langle x_a| = \int \delta(x-x_a)\,\delta(y-x_a)\,dx_a = \delta(x-y) = I$$

identity

$$\boxed{\sum_{x_a} |x_a\rangle\langle x_a| = I}$$

H is independent of time.

The evolution operator reads

$$U(t_b, t_a) = e^{-i\hat{H}(t_b - t_a)}, \quad \boxed{t_b > t_a}$$

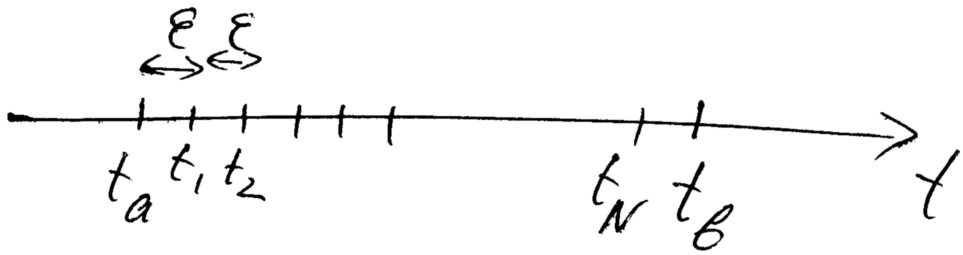
If we know U, we can calculate the probability amplitud to go from a state φ_i at $t = t_a$ to state φ_f at $t = t_b$

$$A = \langle \varphi_f | U(t_b, t_a) | \varphi_i \rangle$$

If φ_i and φ_f are eigenstates of the position operator then A is called propagator (Green's function)

$$K(x_b, t_b, x_a, t_a) = \langle x_b | U(t_b - t_a) | x_a \rangle =$$

$$= \langle x_b | e^{-iH(t_b - t_a)} | x_a \rangle$$



split time interval $[t_a, t_b]$ in $N+1$ equal intervals

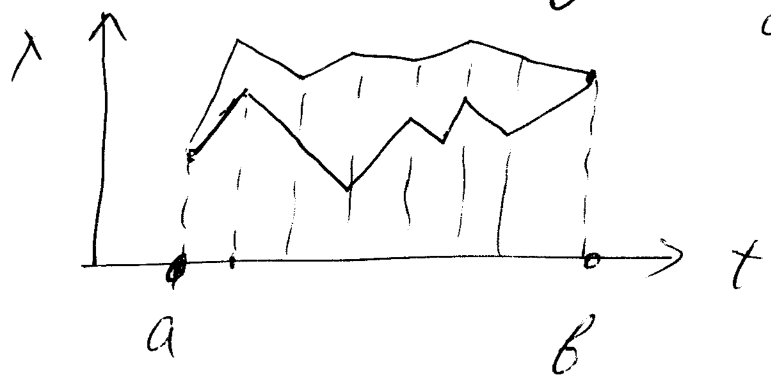
$$\varepsilon = \frac{t_b - t_a}{N+1}$$

$$U(t_b, t_a) = e^{-iH(\overbrace{t_b - t_N}^{\varepsilon})} \underbrace{e^{-iH(\overbrace{t_N - t_{N-1}}^{\varepsilon})}}_{I} \underbrace{\cdots}_{I} e^{-iH(\overbrace{t_1 - t_a}^{\varepsilon})}$$

Use completeness relation $I = \int dx_n |x_n\rangle\langle x_n|$

$$K = \int dx_1 dx_2 \ldots dx_N \, \langle x_b | e^{-i\varepsilon H} | x_N \rangle \ldots \langle x_1 | e^{-i\varepsilon H} | x_a \rangle$$

Physical meaning: propagation along any trajectory is possible.



ε is small, hence

$$\langle x_n | e^{-i\varepsilon H} | x_{n-1} \rangle = \langle x_n | 1 - i\varepsilon H | x_{n-1} \rangle =$$

$$= \delta(x_n - x_{n-1}) - i\varepsilon \langle x_n | H | x_{n-1} \rangle$$

$$H = \frac{\hat{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x), \quad \boxed{\hbar = 1}$$

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momentum representation

$$\delta(x_n - x_{n-1}) = \int e^{ip(x_n - x_{n-1})} \frac{dp}{2\pi}$$

$$|x_n\rangle = \delta(x - x_n) = \int e^{ip(x-x_n)} \frac{dp}{2\pi}$$

$$\left[\langle x_n | \hat{p}^2 | x_{n-1} \rangle = \int dx\, \delta(x - x_n) \left(-\frac{\partial^2}{\partial x^2} \right) \times \right.$$

$$\left. \times \int e^{ip(x - x_{n-1})} \frac{dp}{2\pi} = \int p^2 e^{ip(x_n - x_{n-1})} \frac{dp}{2\pi} \right.$$

$$\left[\langle x_n | V | x_{n-1} \rangle = \int \delta(x - x_n) V(x)\, \delta(x - x_{n-1}) dx = \right.$$

$$\left. = V(x_n)\, \delta(x_n - x_{n-1}) = \int V(x_n) e^{ip(x_n - x_{n-1})} \frac{dp}{2\pi} \right.$$

Altogether

$$\langle x_n | e^{-i\varepsilon H} | x_{n-1} \rangle = \delta(x_n - x_{n-1}) - i\varepsilon \langle x_n | H | x_{n-1} \rangle =$$

$$= \int \frac{dp}{2\pi} e^{ip(x_n - x_{n-1})} \left[1 - i\varepsilon H(p, x_n) \right] \approx$$

$$\approx \int \frac{dp}{(2\pi)} e^{ip(x_n - x_{n-1}) - i\varepsilon H} = \int \frac{dp}{(2\pi)} e^{i\varepsilon[p\dot{x}_n - H(p, x_n)]}$$

Here I take into account that $\dot{x}_n = \frac{x_n - x_{n-1}}{\varepsilon}$

$H(p, x_n)$ is usual <u>Classical</u> Hamiltonian.

Thus

$$K(x_b, t_b, x_a t_a) = \lim_{N\to\infty} \int \prod_{n=1}^{N} dx_n \prod_{n=1}^{N+1} \frac{dp_n}{2\pi} \times$$

$$\times\, e^{i\varepsilon \sum_{n=1}^{N+1} [p_n \dot{x}_n - H(p_n x_n)]}$$

Standard notation

$$\boxed{K = \int \mathcal{D}x\, \mathcal{D}p \exp\left[i \int_{t_a}^{t_b} (p\dot{x} - H(p,x))\, dt \right]}$$

Integration over p can be performed in general case.

$$\int \frac{dp_n}{2\pi} e^{i\varepsilon\left(p_n \dot{x}_n - \frac{p_n^2}{2m}\right)} \longrightarrow \lim_{\alpha\to 0} \int \frac{dp_n}{2\pi} e^{i\varepsilon\left(p_n \dot{x}_n - \frac{p_n^2}{2m}\right) - \alpha p_n^2}$$

We need α to make the integral convergent.

$$i\varepsilon\left(p_n\dot{x}_n - \frac{p_n^2}{2m}\right) - \alpha p_n^2 =$$

$$= i\varepsilon\left[\frac{m\dot{x}_n^2}{2} - \frac{(p_n - m\dot{x}_n)^2}{2m}\right] - \alpha p_n^2 \simeq$$

$$\simeq \underline{i\varepsilon\frac{m\dot{x}_n^2}{2} - \left(\alpha + \frac{i\varepsilon}{2m}\right)(p_n - m\dot{x}_n)^2}$$

$$\int_{-\infty}^{+\infty} e^{-\beta x^2}\,dx = \sqrt{\frac{\pi}{\beta}}$$

$$\int e^{i\varepsilon\frac{m\dot{x}_n^2}{2} - \left(\alpha + \frac{i\varepsilon}{2m}\right)(p_n - m\dot{x}_n)^2}\,\frac{dp_n}{2\pi} =$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{\underset{\substack{\downarrow \\ 0}}{\alpha} + \frac{i\varepsilon}{2m}}}\; e^{i\varepsilon\frac{m\dot{x}_n^2}{2}} = \sqrt{\frac{m}{2\pi i\varepsilon}}\; e^{i\varepsilon\frac{m\dot{x}_n^2}{2}}$$

Thus

$$K = \lim_{N\to\infty}\left(\frac{m}{2\pi i\varepsilon}\right)^{\frac{N+1}{2}}\int\prod_{n=1}^{N} dx_n\, e^{i\varepsilon\sum_{n=1}^{N}\underbrace{\left(\frac{m\dot{x}_n^2}{2} - V(x_n)\right)}_{\text{Lagrangian}}}$$

$$i\varepsilon \sum_{n=1}^{N} L_n = \int_{t_a}^{t_b} L(x,\dot{x})\,dt = S - \text{action}.$$

$$\boxed{K = \int \mathcal{D}x\, e^{iS[x]}}$$

integration over all trajectories. $S[x]$ depends on the trajectory $x(t)$.

The measure $\mathcal{D}x$ in the space of trajectories is defined as

$$\mathcal{D}x \longrightarrow \lim_{N\to\infty} \left(\frac{m}{2\pi i\varepsilon}\right)^{\frac{N+1}{2}} \prod_{n=1}^{N} dx_n$$