PART III

Classical charged particle in an electromagnetic field.

Gauge invariance in quantum case.

Velocity operator.

Landau levels, gauge $(A_x, 0, 0)$

Degeneracy of Landau levels.

Landau levels, "cylindrical gauge".

Charged particle in an electromagnetic field

Classical physics

$\vec{B} = \vec{\nabla} \times \vec{A}$ magnetic field

$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ electric field

Lagrangian of a charged particle

$$\boxed{L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - q\varphi(r) + \frac{q}{c}\dot{\vec{r}}\vec{A}}$$

CGS units

$$\frac{\partial L}{\partial r_i} = -q\frac{\partial \varphi}{\partial r_i} + \frac{q}{c}\dot{r}_j \frac{\partial A_j}{\partial r_i}$$

$$\frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i + \frac{q}{c}A_i$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} = m\ddot{r}_i + \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}\frac{\partial A_i}{\partial r_j}\dot{r}_j$$

The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r_i} = \frac{\partial L}{\partial r_i}$$

$$m\ddot r_i = -q\frac{\partial \varphi}{\partial r_i} + \frac{q}{c}\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)\dot r_j - \frac{q}{c}\frac{\partial A_i}{\partial t}$$

$$\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)\dot r_j = (\vec V \times B)_i$$

$$(V\times B)_i = \varepsilon_{ikl} V_k B_l = \varepsilon_{ikl} V_k \varepsilon_{lmn} \partial_m A_n =$$

$$= (\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}) V_k \partial_m A_n =$$

$$= V_k \partial_i A_k - V_k \partial_k A_i = V_k(\partial_i A_k - \partial_k A_i)$$

$$\boxed{m\ddot r_i = qE_i + \frac{q}{c}[V\times B]_i}$$

Lorentz force

Hamiltonian

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i + \frac{q}{c}A_i \Longleftrightarrow \vec{p} = m\vec{v} + \frac{q}{c}\vec{A}$$

$$H = \vec{p}\cdot\dot{\vec{r}} - L = \left(m\vec{v} + \frac{q}{c}\vec{A}\right)\vec{v} -$$

$$-\frac{1}{2}mv^2 + q\varphi - \frac{q}{c}\vec{v}\vec{A} =$$

$$= \frac{1}{2}mv^2 + q\varphi = \underline{\frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\varphi}$$

Hamilton eqs.

$$\begin{cases} \dot{\vec{r}} = \dfrac{\partial H}{\partial \vec{p}} \\[2ex] \dot{\vec{p}} = -\dfrac{\partial H}{\partial \vec{r}} \end{cases}$$

$$\dot{r}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(p_i - \frac{q}{c}A_i\right) \qquad \text{OK}$$

$$\dot{p}_i = -\frac{\partial H}{\partial r_i} = +\frac{1}{m}\left(p_j - \frac{q}{c}A_j\right)\frac{\partial A_j}{\partial r_i}\frac{q}{c} - q\frac{\partial \varphi}{\partial r_i} =$$

$$= \frac{q}{c}v_j\frac{\partial A_j}{\partial r_i} - q\frac{\partial \varphi}{\partial r_i}$$

$$m\ddot{\vec{r}} = \dot{\vec{p}} - \frac{q}{c}\frac{d\vec{A}}{dt}$$

$$m\ddot{r}_i = \frac{q}{c} v_j \frac{\partial A_j}{\partial r_i} - q\frac{\partial \varphi}{\partial r_i} - \frac{q}{c}\frac{dA_i}{dt}$$

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial r_j}\dot{r}_j$$

$$m\ddot{r}_i = \underbrace{-q\frac{\partial \varphi}{\partial r_i} - \frac{q}{c}\frac{\partial A_i}{\partial t}}_{qE_i} + \frac{q}{c}\underbrace{\left(v_j\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}v_j\right)}_{(V\times B)_i}$$

$$\boxed{m\ddot{\vec{r}} = q\vec{E} + \frac{q}{c}\vec{V}\times\vec{B}}$$ OK

Gauge invariance in classical physics

$$\vec{A} \longrightarrow \vec{A}' = \vec{A} - \vec{\nabla} g$$

$$\varphi \rightarrow \varphi' = \varphi + \frac{1}{c}\frac{\partial g}{\partial t}$$

$$\vec{E} \rightarrow \vec{E}' = -\vec{\nabla}\varphi' - \frac{1}{c}\frac{\partial A'}{\partial t} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial A}{\partial t} = \vec{E}$$

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} - \underbrace{\vec{\nabla}\times\vec{\nabla} g}_{=0} = \vec{\nabla}\times\vec{A} = \vec{B}$$

Field are not changed.

Quantum mechanics.
A charged particle in a static magnetic field.

$$\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}\right)^2$$

$$\hat{\vec{p}} = -i\hbar\vec{\nabla}$$

Schroedinger eq.

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2\psi} \quad (A)$$

gauge transformation of vector potential

$$\vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}g$$

<u>Schroedinger eq is changed under gauge transformation hence ψ is also changed!</u>

Theorem: gauge transformation of the wave function reads.

$$\boxed{\psi' = e^{-i\frac{q}{\hbar c}g(r)}\psi}$$

ψ' obeys Schroedinger eq. in the new gauge.

$$\boxed{i\hbar\frac{\partial\psi'}{\partial t} = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}'\right)^2\psi'} \quad (B)$$

so $\vec{A} \to \vec{A}'$
$\psi \to \psi'$

Let us prove that (A) and (B) are equivalent.

$$\left(\bar{p} - \frac{q}{c}\vec{A}'\right)\psi' = \left(-i\hbar\vec{\nabla} - \frac{q}{c}A + \frac{q}{c}\vec{\nabla}g\right)e^{-i\frac{q}{\hbar c}g}\psi =$$

$$= -i\hbar\vec{\nabla}\left(e^{-i\frac{q}{\hbar c}g}\psi\right) + e^{-i\frac{q}{\hbar c}g}\left(-\frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla}g\right)\psi$$

$$= e^{-i\frac{q}{\hbar c}g}\left\{\cancel{-\frac{q}{c}\vec{\nabla}g\psi} - i\hbar\vec{\nabla}\psi - \frac{q}{c}\vec{A}\psi + \cancel{\frac{q}{c}\vec{\nabla}g\psi}\right\} =$$

$$= e^{-i\frac{q}{\hbar c}g}\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}\right)\psi$$

Similarly

$$\left(\vec{p} - \frac{q}{c}\vec{A}\right)\left[e^{-i\frac{q}{\hbar c}g}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\psi\right] =$$

$$= e^{-i\frac{q}{\hbar c}g}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2\psi$$

Thus eq. B is transformed to

$$\boxed{i\hbar e^{-i\frac{q}{\hbar c}g}\frac{\partial\psi}{\partial t} = e^{-i\frac{q}{\hbar c}g}\frac{\left(\hat{p} - \frac{q}{c}\vec{A}\right)^2}{2m}\psi}$$

Hence it is equivalent to eq. A.

Velocity operator

$$\hat{\vec{V}} = \dot{\vec{r}} = \frac{1}{m}\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}\right)$$

Commutation of different components of velocity operator.

$$[V_i, V_j] = \frac{1}{m^2}\left[p_i - \frac{q}{c}A_i, p_j - \frac{q}{c}A_j\right] =$$

$$= \frac{1}{m^2}\left\{[p_i, p_j] + \frac{q^2}{c^2}[A_i, A_j] - \right.$$

$$\left. - \frac{q}{c}[A_i, p_j] - \frac{q}{c}[p_i, A_j]\right\} =$$

$$= -\frac{q}{m^2 c}(-i\hbar)\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right) = \frac{i\hbar q}{m^2 c}\varepsilon_{ijk}B_k$$

$$\varepsilon_{ijk}B_k = \varepsilon_{ijk}\varepsilon_{kmn}\partial_m A_n =$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_m A_n =$$

$$= \partial_i A_j - \partial_j A_i \quad \text{ok.}$$

$$\boxed{[\hat{V}_n, \hat{V}_m] = i\frac{\hbar q}{m^2 c}\varepsilon_{nml}B_l}$$

uniform field, $B = B_z$

$$[V_x, V_z] = 0$$

$$[V_y, V_z] = 0$$

$$[V_x, V_y] = i\frac{\hbar q}{m^2 c}B.$$

Components of velocity cannot have definite values simultaneously.

if V_z and V_x have definite values then V_y is completely uncertain. $\Delta V_y = \infty$.

if V_z and V_y have definite values then V_x is completely uncertain.

$$\Delta V_x = \infty.$$

Charged particle in an uniform magnetic field. Landau levels.

$\vec{B} = (0, 0, B)$

Let us choose $\vec{A} = (-By, 0, 0)$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$$\left.\begin{aligned} B_x &= \partial_x A_z - \partial_z A_x = 0 \\ B_y &= \partial_z A_x - \partial_x A_z = 0 \\ B_z &= \partial_x A_y - \partial_y A_x = B \end{aligned}\right\} \quad \text{OK.}$$

Hamiltonian

$$\hat{H} = \frac{\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}\right)^2}{2m} = \frac{\left(\hat{p}_x + \frac{q}{c}By\right)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$\left\{\frac{\left(\hat{p}_x + \frac{q}{c}By\right)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}\right\}\psi = \varepsilon\psi$$

Solution

$$\psi = e^{ik_x x + ik_z z}\,\chi(y)$$

k_x and k_z are arbitrary

physical meaning: free propagation along x and z directions, confinement in y-direction.

Substitute ψ in schroedinger eq.

$$\begin{cases} \left(p_x + \frac{q}{c}By\right)\psi = \left(\hbar k_x + \frac{q}{c}By\right)\psi \\ \left(p_x + \frac{q}{c}By\right)^2\psi = \left(\hbar k_x + \frac{q}{c}By\right)^2\psi = \frac{q^2B^2}{c^2}(y-y_0)^2\psi \\ \text{where } y_0 = -\frac{c\hbar k_x}{qB} \end{cases}$$

$$\left\{ p_z^2\psi = \hbar^2 k_z^2\psi \right.$$

Schroedinger eq. is transformed to

$$\left(\frac{q^2B^2}{2mc^2}(y-y_0)^2+\frac{\hbar^2k_z^2}{2m}+\frac{p_y^2}{2m}\right)\chi=\varepsilon\chi$$

$$\boxed{\omega_c=\frac{|q|B}{mc}}$$ — cyclotron frequency.

$$\left[\frac{\hat{p}_y^2}{2m}+\frac{m\omega_c^2}{2}(y-y_0)^2\right]\chi=\underbrace{\left(\varepsilon-\frac{\hbar^2k_z^2}{2m}\right)}_{\varepsilon'}\chi$$

This is 1D harmonic oscillator problem. Solution is known.

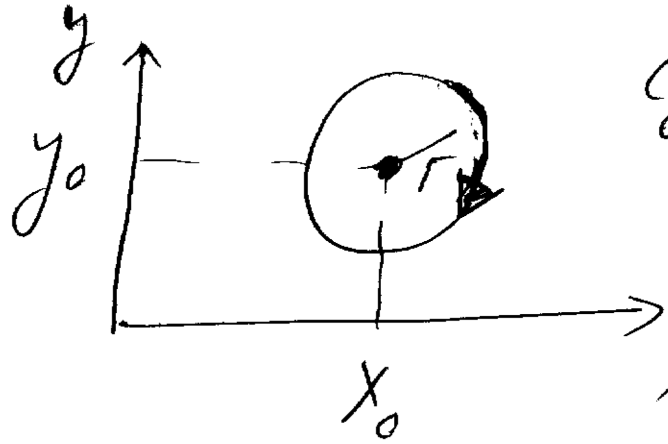
$\varepsilon_n'=\hbar\omega_c(n+\frac{1}{2})$ — Landau levels

$$\boxed{\varepsilon=\varepsilon(n,k_z)=\hbar\omega_c(n+\tfrac{1}{2})+\frac{\hbar^2k_z^2}{2m}}$$

The energy is independent of y_0

$\Longrightarrow$ degeneracy

Classical physics



Cyclotron rotation

$$X = X_0 + r\cos\omega_c t$$

$$\frac{mV^2}{r} = \frac{q}{c} VB \Longrightarrow \frac{mV}{r} = \frac{qB}{c} \Longrightarrow$$

$$\Longrightarrow r = mv\frac{c}{qB} = p\frac{c}{qB}$$

$$X = X_0 - \frac{c}{qB} p_y$$

Quantum

operator of the center of the rotation

$$\boxed{\hat{X}_0 = \hat{X} + \frac{c}{qB}\hat{p}_y}$$

Commutes with Hamiltonian (see page 58)

$$\hat{H} = \frac{\left(p_x + \frac{q}{c}By\right)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$[\hat{X}_0 \hat{H}] = \left[\hat{x} + \frac{c}{qB}\hat{p}_y, \frac{1}{2m}\left(\hat{p}_x + \frac{q}{c}B\hat{y}\right)^2\right] =$$

$$= \frac{1}{2m}\left(\hat{p}_x + \frac{q}{c}B\hat{y}\right)\underbrace{\left[\hat{X} + \frac{c}{qB}\hat{p}_y, \hat{p}_x + \frac{q}{c}B\hat{y}\right]}\Rightarrow$$

$$\rightarrow [\hat{X}, \hat{p}_x] + [\hat{p}_y, \hat{y}] = i\hbar - i\hbar = 0 \quad \text{ok.}$$

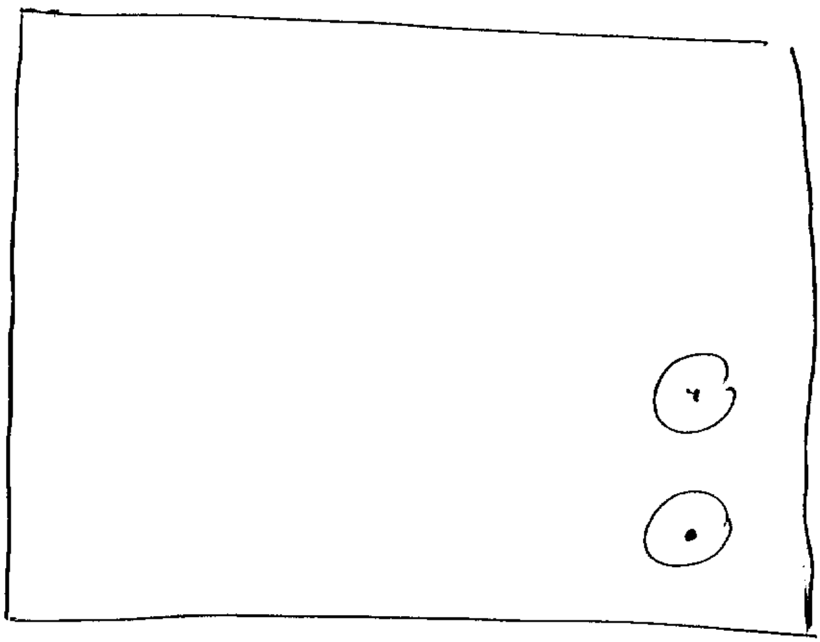
$\hat{X}_0$ does not commute with $\hat{y}_0$

$\hat{y}_0 = -\frac{c}{qB}\hat{p}_x$, see page 59

$$[\hat{y}_0, \hat{X}_0] = \left[-\frac{c}{qB}\hat{p}_x, \hat{X} + \frac{c}{qB}\hat{p}_y\right] = -\frac{c}{qB}[p_x, X] = i\hbar\frac{c}{qB}$$

Hence, since y_0 has a definite value, x_0 is completely uncertain. This is the physical reason for degeneracy.

L_y



L_x

$A = L_x L_y$

$L_x, L_y \to \infty$

Degeneracy of a single Landau level.

$$\Delta N = \frac{L_x \Delta p_x}{2\pi\hbar}$$

$$0 < y_0 < L_y \Rightarrow \Delta p_x = \frac{qB}{c} L_y$$

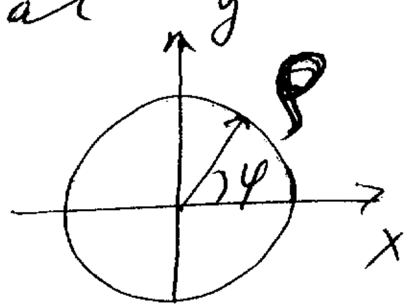
hence $$\Delta N = \frac{qB\, L_x L_y}{2\pi\hbar c} = \frac{qBA}{2\pi\hbar c}$$

Finally Landau levels with account of spin, $s = 1/2$

$$\boxed{\mathcal{E} = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m} + 2\mu S_z B}$$

$S_z = \pm\frac{1}{2}$, μ is magnetic moment of the particle.

Solution in cylindrical coordinates.



Another gauge.

$$\vec{A} = -\frac{1}{2}[\vec{r} \times B] = \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right)$$

compare with page 58

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 = \frac{p^2}{2m} - \frac{q}{mc}\vec{A}\cdot\vec{p} + \frac{q^2}{2mc^2}A^2$$

$$\vec{A}^2 = \frac{1}{4}[\vec{r}\times\vec{B}]^2 = \frac{B^2}{4}(x^2+y^2) = \frac{B^2}{4}\rho^2$$

$$\vec{A}\cdot\vec{p} = -\frac{1}{2}[\vec{r}\times\vec{B}]\vec{p} = -\frac{1}{2}[\vec{p}\times\vec{r}]\vec{B} =$$

$$= \frac{1}{2}\vec{B}\cdot\underbrace{[\vec{r}\times\vec{p}]}_{\hat{\vec{l}}} = \frac{1}{2}\vec{B}\vec{l} = \frac{1}{2}Bl_z = -\frac{i\hbar}{2}B\frac{\partial}{\partial\varphi}$$

$$\frac{p^2}{2m} = -\frac{\hbar^2}{2m}\Delta = -\frac{\hbar^2}{2m}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right\}$$

Schroedinger eq.

$$\hat{H}\psi = \varepsilon\psi$$

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}\right\} + \frac{iq\hbar B}{2mc}\frac{\partial\psi}{\partial\varphi} + \frac{q^2B^2}{8mc^2}\rho^2\psi = \varepsilon\psi$$

$$\psi = R(\rho)e^{iM\varphi}e^{ik_z z}$$

M is an integer number, $M = 0, \pm1, \pm2, \ldots$

Radial Schroedinger eq.

$$\begin{cases} -\frac{\hbar^2}{2m}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} - k_z^2 R - \frac{M^2}{\rho^2}R\right\} - \\ -\frac{q\hbar B M}{2mc}R + \frac{q^2B^2}{8mc^2}\rho^2 R = \mathcal{E}R \end{cases}$$

one can show that the spectrum is exactly the same as in previous gauge

$$\mathcal{E} = \hbar\omega_c\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}$$

but wave functions are different.

Ground state

$\underline{M=0}$

$$R = Ae^{-\alpha\rho^2} \longrightarrow e^{-\alpha\rho^2}$$

$$\frac{\partial R}{\partial \rho} = -2\alpha\rho e^{-\alpha\rho^2}$$

$$\rho\frac{\partial R}{\partial \rho} = -2\alpha\rho^2 e^{-\alpha\rho^2}$$

$$\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial \rho} = -4\alpha\rho e^{-\alpha\rho^2} + 4\alpha^2\rho^3 e^{-\alpha\rho^2}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial \rho} = -4\alpha e^{-\alpha\rho^2} + 4\alpha^2\rho^3 e^{-\alpha\rho^2}$$

$$\underline{\frac{\hbar^2}{2m}4\alpha} - \underline{\underline{\frac{4\alpha^2\hbar^2}{2m}\rho^2}} + \underline{\underline{\frac{q^2B^2}{8mc^2}\rho^2}} = \underline{\varepsilon - \frac{\hbar^2 k_z^2}{2m}}$$

$$\alpha = \frac{|q|B}{4\hbar c}$$

$$\varepsilon - \frac{\hbar^2 k_z^2}{2m} = \frac{\hbar^2}{2m}4\alpha = \frac{\hbar}{2}\frac{qB}{mc} = \frac{\hbar\omega_c}{2} \quad \text{OK!}$$